Material

3 main parts

covered in midterms
- power series & convergence (uniform and pointwise) of functions
- differentiation, Taylor's theorem, l'Hospital's rule
- integration.

Final will be cumulative, somewhat more emphasis on later parts of course differentiation:

<u>Definition:</u> A function $f : (a,b) \to \mathbb{R}$ is differentiable at $x_0 \in (a,b)$

if $\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$ exists

limit $= f'(x_0)$

Consider $f(x) = \begin{cases} 0 & x \text{ rational} \\ x^2 & x \text{ irrational} \end{cases}$

- Is f differentiable at any point $x \in \mathbb{R}$?
- Is f continuous at " "
- Is f integrable on $[0,1]$.

try $x_0 = 0$

need to consider $\lim\limits_{x \to 0} \dfrac{f(x) - \overset{f(0)}{0}}{x - 0} =$

$$= \lim_{x \to 0} \frac{f(x)}{x}$$

$$\frac{f(x)}{x} = \begin{cases} 0 & x \text{ rational} \\ x & x \text{ irrational} \end{cases}$$

formal proof: $\varepsilon > 0$

$\Rightarrow$ if $|x - 0| = |x| < \varepsilon$

$$\Rightarrow \left| \frac{f(x)}{x} - 0 \right| = \left. \begin{cases} |0 - 0| & x \text{ rat.} \\ |x - 0| & x \text{ irrat.} \end{cases} \right\} \underline{\leq} |x| < \varepsilon$$

in both cases

$$\Rightarrow 0 \le \lim_{x\to 0} \left| \frac{f(x)}{x} - 0 \right| \le \lim_{x\to 0} |x| = 0$$

$$\Rightarrow \quad f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = 0$$

claim: $x = 0$ is the only point where f is continuous
($\Leftrightarrow$ only point " " " differentiable)

If $x_0 \neq 0$

- can find sequence of rational numbers $x_n \to x_0$

$$\Rightarrow \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0$$

- can find sequence of irrational numbers $\tilde{x}_n \to x_0$

$$\Rightarrow \lim_{n\to\infty} f(\tilde{x}_n^2) = \lim_{n\to\infty} \tilde{x}_n^2 = x_0^2$$

$\Rightarrow \lim_{x \to x_0} f(x)$ does not exist for $x_0 \neq 0$

- integrability on $[0,1]$?

need to show:

$$U(f) = \inf_P \{U(f,P)\} = L(f) = \sup_P \{L(f,P)\}$$

claim: $L(f) = 0$

proof. for any $t_{k-1} < t_k$ $\exists$ rational number $x_k \in (t_{k-1}, t_k)$

$\Rightarrow \inf \{f(x), x \in (t_{k-1}, t_k)\} \leq 0$

in fact as $f(x) = x^2 \geq 0$ for x irrational

we can write $=$

$= m(f, [t_{k-1}, t_k])$

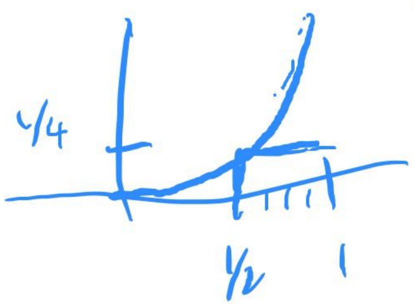
$\Rightarrow$ for any partition $P = \{\underset{0}{\underset{\shortparallel}{t_0}} < t_1 < \dots < t_n = 1\}$

we have $L(f,P) = \sum_{k=1}^{n} \underbrace{m(f,(t_{k-1},t_k))}_{=0}(t_k - t_{k-1})$

$= 0$

<u>Claim:</u> For any partition P $\exists$ partition $\tilde{P}$ s.t.

$U(f,P) > 1/8$



$y = x^2$

idea: add point $1/2$ to P

$\tilde{P} = P \cup \{1/2\}$

$\Rightarrow$ if $1/2 \le t_{k-1} < t_k < 1$

$\Rightarrow$ $\sup\{ f(x),\ x \in [t_{k-1}, t_k]\} \ge t_{k-1}^2 \ge 1/4$

reason: $\exists$ irrational number $y \in (t_{k-1}, t_k)$

s.t. $y > t_{k-1} \ge 1/2$

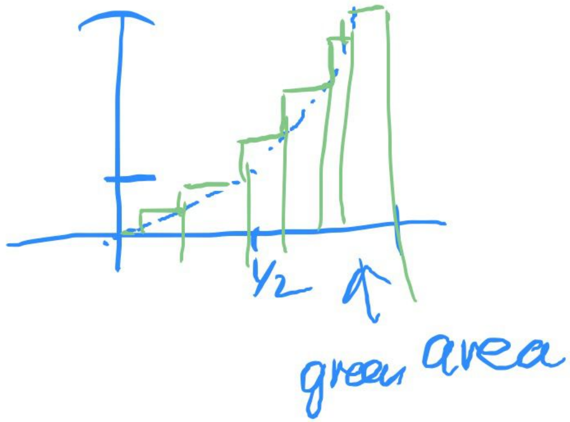
$\Rightarrow$ $f(y) = y^2 \ge (1/2)^2 \ge 1/4.$ $M(f, [t_{k-1}, t_k]) \ge 1/4.$

$$\Rightarrow U(f, \tilde{P}) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1})$$

$$\ge \sum_{\substack{k \\ t_{k-1} \ge 1/2}} \frac{1}{4} (t_k - t_{k-1})$$

$$= \frac{1}{4} \sum_{t_{k-1} \ge 1/2} (t_k - t_{k-1}) = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$$

pictorially



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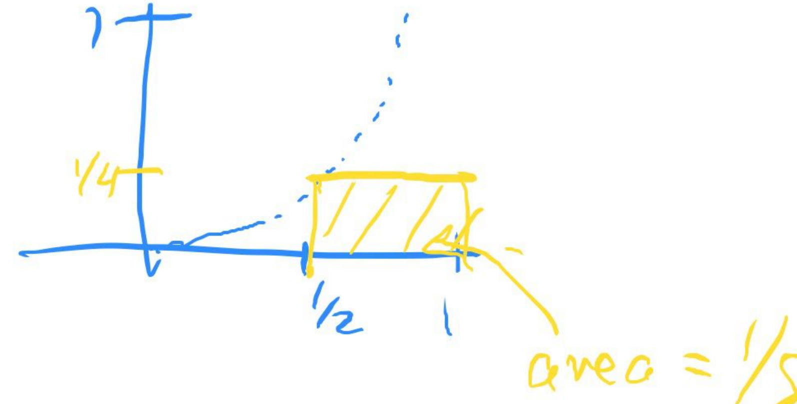


Conclusion:

$$U(f) = \inf_{P \text{ partition}} (f,P) \geq 1/4 > 0 = L(f,P)$$

$\Rightarrow$ not integrable.

midterm 2 question:

show:

$$1 + \frac{x}{2} - \frac{x^2}{8} < \sqrt{1+x} < 1 + \frac{x}{2}$$

idea for solution: use Taylor's Theorem.

$$\sqrt{1+x} = \underbrace{1 + \frac{x}{2}}_{\text{linear Taylor poly}} + R_2(x)$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + R_3(x)$$

use Taylor's theorem to show that $R_2(x) < 0$ and $R_3(x) > 0$ $\Big\} \Rightarrow$ claim.

T/F question in midterm 1

If $f_n \to f$ pointwise on $[0,1]$

$$\Longrightarrow \int_0^1 f_n(x)\,dx \longrightarrow \int_0^1 f(x)\,dx$$

statement is false !

<u>example 1</u>: $f_n(x) = (n+1)x^n$

obviously. $\boxed{\int_0^1 f_n(x)\,dx = x^{n+1}\Big|_0^1 = 1}$

fix $0 < x < 1$ let $a_n = (n+1)x^n$

$$\Longrightarrow \frac{a_{n+1}}{a_n} = \frac{n+1}{n} x$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{n} x = x < 1$$

$$\Rightarrow \lim_{n\to\infty} a_n = 0$$ (part of 142A)

(reason: can deduce from this:

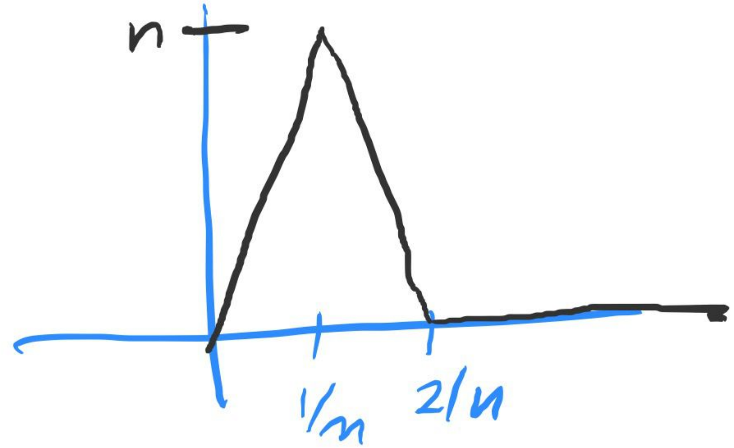
$$|a_n| \leq M\left(\frac{x+1}{2}\right)^n$$

$$\frac{x+1}{2} < 1$$

$$\Rightarrow \lim a_n = 0.$$

Another example:

$f_n(x)$ given by



again: $\int_0^1 f_n(x)\,dx = 1$

and $f_n(x) \to 0$ for $n \to \infty$

in both cases: $f_n \to 0 = f$ pointwise

but $\boxed{\int_0^1 f_n(x)\,dx = 1 \neq \int_0^1 f(x)\,dx = 0}$